Name:

Smart Grids Technologies
Exercise
Per-Unit Calculus

1 Organization

This exercise session serves as preparation for Labs 2.1 – 2.3 where you will work mainly with per-unit quantities. Sec. 2 revises the theory concerning the per-unit analysis of power systems. Sec. 3 contains some simple exercises for practicing what you have learnt. **You do not have to upload this report on moodle.** A sample solution will be provided so that you can check your answers.

2 Theory

2.1 Introduction

A *per-unit system* is obtained by referring all quantities of a system to certain *base values*. Consider a complex quantity $\bar{A} = A_{re} + jA_{im}$ (i.e., $A_{re} = \Re\{\bar{A}\}$, $A_{im} = \Im\{\bar{A}\}$). Let $A_b \in \mathbb{R}^+$ be the corresponding base value, which applies to both the real and the imaginary part. The corresponding per-unit quantity $\bar{a}$ is obtained as follows

$$\bar{a} = \frac{\bar{A}}{A_b} = \frac{A_{re} + jA_{im}}{A_b} = \frac{A_{re}}{A_b} + j\frac{A_{im}}{A_b} \tag{1}$$

Accordingly, for the absolute value $|\bar{a}|$

$$|\bar{a}| = \sqrt{a_{re}^2 + a_{im}^2} = \sqrt{\left(\frac{A_{re}}{A_b}\right)^2 + \left(\frac{A_{im}}{A_b}\right)^2} = \frac{\sqrt{A_{re}^2 + A_{im}^2}}{A_b} = \frac{|\bar{A}|}{A_b} \tag{2}$$

One says that $\bar{A}$ is given in *absolute* units, and $\bar{a}$ in *relative* (or *normalized*) units. The base units for different quantities have to be *coherent*. A set of base units $\{A_{b,k} \mid k \in \mathcal{K}\})$ is coherent if the following holds: if the quantity A_i ($i \in \mathcal{K}$) depends on the other quantities A_j ($j \in \mathcal{K}$, $j \neq i$) according to a physical law f_i of the form

$$A_i = f_i\left(\{A_j \mid j \in \mathcal{K},\ j \neq i\}\right) \tag{3}$$

the corresponding base units are linked by the same physical law. That is

$$\forall i \in \mathcal{K}:\ A_{b,i} = f_i\left(\{A_{b,j} \mid j \in \mathcal{K},\ j \neq i\}\right) \tag{4}$$

In power system analysis, one needs to define base values for voltage, current, power, and impedance (or admittance). Obviously, these quantities are not mutually independent. Usually, one chooses base values for power and voltage, and then computes the remaining ones from these.

2.2 Single-Phase Systems

Consider a single-phase system. Select base values V_b for the voltage and A_b for the power (i.e., P or Q). The base values for current and impedance (or admittance) are obtained as follows

$$I_b = \frac{A_b}{V_b} \tag{5}$$

$$Z_b = \frac{V_b}{I_b} = \frac{V_b^2}{A_b} \quad \left(Y_b = \frac{1}{Z_b} = \frac{I_b}{V_b} = \frac{A_b}{V_b^2}\right) \tag{6}$$

Let $\bar{V}$ and $\bar{I}$ be the phasors of voltage and current, and $\bar{S} = \bar{V}\bar{I}^* = P + jQ$ be the complex power in absolute units at a given point of the grid. The corresponding per-unit quantities $\bar{v}$, $\bar{i}$, and $\bar{s} = p + jq$ are obtained as follows

$$\bar{v} = \frac{\bar{V}}{V_n} \tag{7}$$

$$\bar{i} = \frac{\bar{I}}{I_b} \tag{8}$$

$$\bar{s} = \frac{\bar{S}}{A_b} \quad \left(p = \frac{P}{A_b},\ q = \frac{Q}{A_b}\right) \tag{9}$$

Let $\bar{Z}$ be the impedance in absolute values of an element in the grid. The corresponding per-unit quantity $\bar{z}$ is given by

$$\bar{z} = \frac{\bar{Z}}{Z_b} = \bar{Z}\frac{I_b}{V_b} = \bar{Z}\frac{A_b}{V_b^2} \quad \left(\bar{y} = \frac{\bar{Y}}{Y_b} = \bar{Y}\frac{V_b}{I_b} = \bar{Y}\frac{V_b^2}{A_b}\right) \tag{10}$$

2.3 Three-Phase Systems

As explained above, in single-phase systems, the base values A_b and V_b correspond to the power and voltage of one phase. In principle, the base values can be defined analogously for three-phase systems. However, the data of three-phase components are typically given in terms of *three-phase power* and *phase-to-phase voltage*. Therefore, one usually chooses A_b as the three-phase power and V_b as the phase-to-phase voltage. By consequence, the base values for current and impedance (or admittance) are given by

$$I_b = \frac{A_b}{\sqrt{3}V_b} \tag{11}$$

$$Z_b = \frac{V_b^2}{A_b} \quad \left(Y_b = \frac{A_b}{V_b^2}\right) \tag{12}$$

If the three-phase system is perfectly *balanced* and in a sinusoidal regime of operation, it can be reduced to an equivalent single-phase system (i.e., the *positive-sequence* system). The voltages in this equivalent system are phase-to-ground voltages. The relation between the phase-to-phase base voltage V_b and the phase-to-ground base voltage E_b is

$$E_b = \frac{V_b}{\sqrt{3}} \tag{13}$$

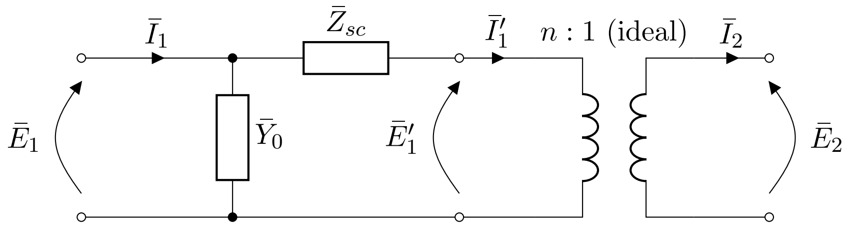


Figure 1: Equivalent circuit of a transformer in absolute units.

The values of the phase-to-phase voltage relative to V_b and of the phase-to-ground voltage relative to E_b in a point of the system are equal. That is

$$\bar{v} = \frac{\bar{V}}{V_b} = \frac{\sqrt{3}\bar{E}}{\sqrt{3}E_b} = \frac{\bar{E}}{E_b} = \bar{e} \tag{14}$$

2.4 Change of Base

Let $\bar{v}_1$, $\bar{i}_1$, $\bar{s}_1$, $\bar{z}_1$ and $\bar{y}_1$ be values in per unit of the base defined by $A_{b,1}$, $V_{b,1}$. Similarly, let $\bar{v}_2$, $\bar{i}_2$, $\bar{s}_2$, $\bar{z}_2$ and $\bar{y}_2$ be the same quantities but with respect to another base defined by $A_{b,2}$, $V_{b,2}$. The following identities hold

$$\frac{\bar{v}_1}{\bar{v}_2} = \frac{\bar{V}}{V_{b,1}} \left(\frac{\bar{V}}{V_{b,2}}\right)^{-1} = \frac{V_{b,2}}{V_{b,1}} \tag{15}$$

$$\frac{\bar{i}_1}{\bar{i}_2} = \frac{\bar{I}}{I_{b,1}} \left(\frac{\bar{I}}{I_{b,2}}\right)^{-1} = \frac{I_{b,2}}{I_{b,1}} = \frac{A_{b,2}}{A_{b,1}} \frac{V_{b,1}}{V_{b,2}} \tag{16}$$

$$\frac{\bar{s}_1}{\bar{s}_2} = \frac{\bar{S}}{A_{b,1}} \left(\frac{\bar{S}}{A_{b,2}}\right)^{-1} = \frac{A_{b,2}}{A_{b,1}} \tag{17}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{\bar{Z}}{Z_{b,1}} \left(\frac{\bar{Z}}{Z_{b,2}}\right)^{-1} = \frac{Z_{b,2}}{Z_{b,1}} = \left(\frac{V_{b,2}}{V_{b,1}}\right)^2 \frac{A_{b,1}}{A_{b,2}} \tag{18}$$

$$\frac{\bar{y}_1}{\bar{y}_2} = \frac{\bar{z}_2}{\bar{z}_1} = \left(\frac{V_{b,1}}{V_{b,2}}\right)^2 \frac{A_{b,2}}{A_{b,1}} \tag{19}$$

2.5 Transformer Equivalent Circuit

Fig. 1 depicts the single-phase equivalent circuit of a three-phase transformer with star configuration on both sides (i.e., on primary and secondary side). Let $V_{n,1}$ and $V_{n,2}$ be the nominal voltage on the primary and secondary side and $n = V_{n,1}/V_{n,2}$ the *nominal transformer ratio*, where

$$\bar{E}_1' = n\bar{E}_2 \tag{20}$$

$$\bar{I}_1' = \frac{1}{n}\bar{I}_2 \tag{21}$$

$\bar{Y}_0 = G_0 + jB_0$ is the *zero-load admittance* referred to the primary side, and $\bar{Z}_{sc} = R_{sc} + jX_{sc}$ is the *short-circuit impedance* referred to the primary side. The transformer is described by the following equations

$$\bar{E}_1 = n\bar{E}_2 + \frac{\bar{Z}_{sc}}{n}\bar{I}_2 \tag{22}$$

$$\bar{I}_1 = \bar{Y}_0\bar{E}_2 + \frac{1}{n}(1 + \bar{Y}_0\bar{Z}_{sc})\bar{I}_2 \tag{23}$$

Fix a base value A_b for the three-phase power, and two base values $V_{b,1}$ and $V_{b,2}$ for the phase-to-phase voltage on the primary and secondary side. In the per-unit system, the electrical parameters are given by

$$\bar{z}_{sc} = \bar{Z}_{sc}\frac{A_b}{V_{b,1}^2} \tag{24}$$

$$\bar{y}_0 = \bar{Y}_0\frac{V_{b,1}^2}{A_b} \tag{25}$$

Accordingly, the transformer equations (22) & (23) become

$$\bar{v}_1 = \bar{E}_1\frac{\sqrt{3}}{V_{b,1}} \tag{26}$$

$$= n\bar{E}_2\frac{\sqrt{3}}{V_{b,1}}\frac{V_{b,2}}{V_{b,2}} + \frac{\bar{Z}_{sc}}{n}\bar{I}_2\frac{\sqrt{3}}{V_{b,1}} \tag{27}$$

$$= n\bar{v}_2\frac{V_{b,2}}{V_{b,1}} + \frac{\bar{z}_{sc}}{n}\bar{i}_2\frac{V_{b,1}}{V_{b,2}} \tag{28}$$

$$\bar{i}_1 = \bar{I}_1\frac{\sqrt{3}V_{b,1}}{A_b} \tag{29}$$

$$= n\bar{Y}_0\bar{E}_2\frac{\sqrt{3}V_{b,1}}{A_b}\frac{V_{b,2}}{V_{b,2}}\frac{V_{b,1}}{V_{b,1}} + \frac{1}{n}(1 + \bar{Y}_0\bar{Z}_{sc})\bar{I}_2\frac{\sqrt{3}V_{b,1}}{A_b}\frac{V_{b,2}}{V_{b,2}} \tag{30}$$

$$= n\bar{y}_0\bar{v}_2\frac{V_{b,2}}{V_{b,1}} + \frac{1}{n}(1 + \bar{y}_0\bar{z}_{sc})\bar{i}_2\frac{V_{b,1}}{V_{b,2}} \tag{31}$$

If the base voltages are chosen such that their ratio is equal to the transformer ratio, that is

$$\frac{V_{b,1}}{V_{b,2}} = n \tag{32}$$

the per-unit transformer equations (28)–(31) simplify to

$$\bar{v}_1 = \bar{v}_2 + \bar{z}_{sc}\bar{i}_2 \tag{33}$$

$$\bar{i}_1 = \bar{y}_0\bar{v}_2 + (1 + \bar{y}_0\bar{z}_{sc})\bar{i}_2 \tag{34}$$

Accordingly, the transformer is described by the following system of linear equations (i.e., $ABCD$ parameters)

$$\begin{bmatrix} \bar{v}_1 \\ \bar{i}_1 \end{bmatrix} = \begin{bmatrix} 1 & \bar{z}_{sc} \\ \bar{y}_0 & 1 + \bar{y}_0\bar{z}_{sc} \end{bmatrix} \begin{bmatrix} \bar{v}_2 \\ \bar{i}_2 \end{bmatrix} \tag{35}$$

This corresponds to the equivalent circuit shown in Fig. 2.

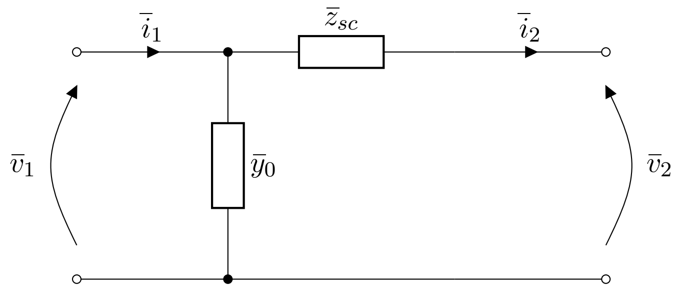


Figure 2: Equivalent circuit of a transformer in relative units when $\frac{V_{b,1}}{V_{b,2}} = n$.

If (32) is not satisfied, that is

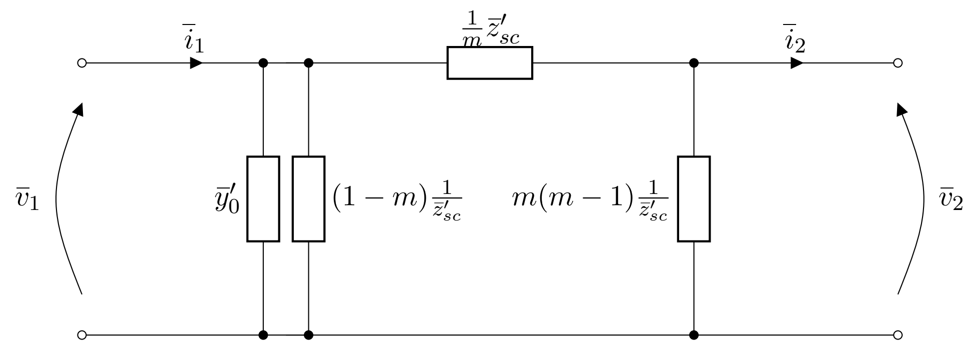
$$\frac{V_{b,1}}{V_{b,2}} \neq n \tag{36}$$

we define the *per-unit transformer ratio* m as

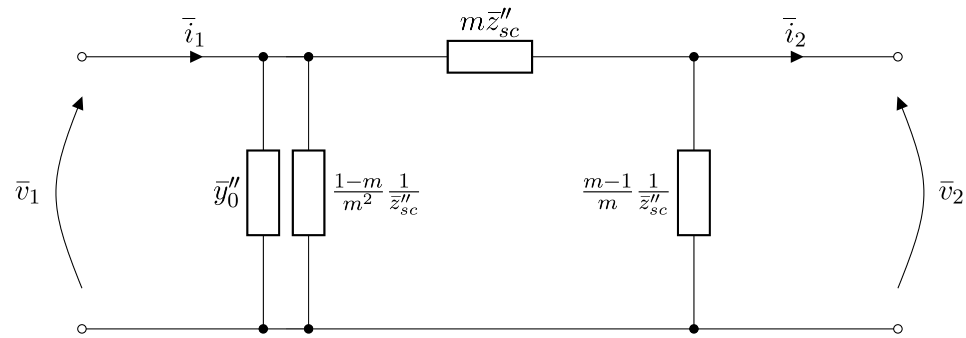
$$m = n\left(\frac{V_{b,1}}{V_{b,2}}\right)^{-1}. \tag{37}$$

The transformer equations (35) become

$$\begin{bmatrix} \bar{v}_1 \\ \bar{i}_1 \end{bmatrix} = \begin{bmatrix} m & \frac{1}{m}\bar{z}'_{sc} \\ m\bar{y}'_0 & \frac{1}{m}(1 + \bar{y}'_0\bar{z}'_{sc}) \end{bmatrix} \begin{bmatrix} \bar{v}_2 \\ \bar{i}_2 \end{bmatrix} \tag{38}$$



(a) Referred to the primary side.



(b) Referred to the secondary side.

Figure 3: Per-unit equivalent circuit of a transformer when $\frac{V_{b,1}}{V_{b,2}} \neq n$.

where $\bar{z}'_{sc}$ and $\bar{y}'_0$ are the per-unit values of $\bar{Z}_{sc} = \bar{Z}'_{sc}$ and $\bar{Y}_0 = \bar{Y}'_0$ referred to the primary side. This yields the per-unit equivalent circuit shown in Fig. 3a. Analogously, let $\bar{z}''_{sc}$ and $\bar{y}''_0$ be the per-unit values of $\bar{Z}_{sc} = \bar{Z}''_{sc}$ and $\bar{Y}_0 = \bar{Y}''_0$ referred to the secondary side. Then, the corresponding per-unit equivalent circuit is shown in Fig. 3b. This can be easily derived by knowing that $\bar{Z}''_{sc} = \frac{1}{n^2}\bar{Z}'_{sc}$.

Typically, the following quantities are specified in transformer data sheets: the *short-circuit voltage* $V_{sc,\%}$ (in % of the nominal voltage), the *winding losses* $P_{sc,\%}$ (in % of the nominal power), the *core losses* $P_{0,\%}$ (in % of the nominal power), and the *zero-load current* $I_{0,\%}$ (in % of the nominal current). Suppose that the base power and voltages are chosen equal to the nominal power and voltages of the transformer (i.e., $A_b = A_n$, $V_{b,1} = V_{n,1}$, $V_{b,2} = V_{n,2}$). Then, the parameters of the per-unit equivalent circuit can be computed as follows

$$z_{sc} = |\bar{z}_{sc}| = Z_{sc}\frac{A_n}{V_{n,1}^2} = \frac{\frac{V_{n,1}}{\sqrt{3}}\frac{V_{sc,\%}}{100}}{I_{n,1}}\frac{A_n}{V_{n,1}^2} = \frac{V_{sc,\%}}{100} \tag{39}$$

$$r_{sc} = R_{sc}\frac{A_n}{V_{n,1}^2} = \frac{A_n\frac{P_{sc,\%}}{100}}{3I_{n,1}^2}\frac{A_n}{V_{n,1}^2} = \frac{P_{sc,\%}}{100} \tag{40}$$

$$y_0 = |\bar{y}_0| = Y_0\frac{V_{n,1}^2}{A_n} = \frac{I_{n,1}\frac{I_{0,\%}}{100}}{\frac{V_{n,1}}{\sqrt{3}}}\frac{V_{n,1}^2}{A_n} = \frac{I_{0,\%}}{100} \tag{41}$$

$$g_0 = G_0\frac{V_{n,1}^2}{A_n} = \frac{A_n\frac{P_{0,\%}}{100}}{V_{n,1}^2}\frac{V_{n,1}^2}{A_n} = \frac{P_{0,\%}}{100} \tag{42}$$

In the above equations, all parameters are referred to the primary side, but the results are the same if they are referred to the secondary side. The reactance x_{sc} and the susceptance b_0 are obtained as follows

$$x_{sc} = \sqrt{z_{sc}^2 - r_{sc}^2} \tag{43}$$

$$b_0 = \sqrt{y_0^2 - g_0^2} \tag{44}$$

In practice, r_{sc} and $\bar{y}_0$ are often negligible (i.e., $\bar{z}_{sc} = jx_{sc}$ and $\bar{y}_0 = 0$).

2.6 Per-Unit Analysis

Consider a power system with transformers, whose transformation ratios are fixed (i.e., there are no tap-changing transformers). In general, such a system includes subsystems at different voltage levels. The following procedure allows to analyze such systems in a simple manner

1. Fix a unique base power A_b for the entire system, and individual base voltages $V_{b,k}$ for all the subsystem k, such that condition (32) is satisfied for as many as possible transformers. From these, the corresponding base impedances $Z_{b,k}$ (or base admittances $Y_{b,k}$) can be computed.

2. Derive the per-unit equivalent circuits of all electrical components in the power systems (e.g., generators, transformers, lines, and loads), and compose the per-unit model of the entire power system.

3. Formulate the system of equations describing the per-unit model, and solve them to obtain the desired unknowns in relative units.

4. Transform the results from relative to absolute units.

3 Exercises

3.1 Linear Feeder

Consider the linear feeder shown in Fig. 4, which consists of a synchronous machine (SM), two transformers (TF1, TF2), a transmission line (TL), and a load (L). The electrical parameters of these components are listed in Tab. 1.

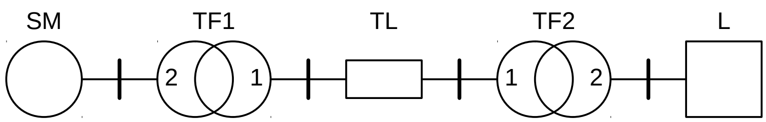


Figure 4: Schematic of the linear feeder.

Table 1: Electrical parameters of the components forming the linear feeder.

Component	Parameters
SM	A_n=50 MVA, V_n=12 kV, x_{SM}=1.1 pu ($\bar{Z}_{\text{SM}} = jX_{\text{SM}}$)
TF1	A_n=60 MVA, $V_{n,1}$=220 kV, $V_{n,2}$=10 kV, $V_{sc,\%}$=10%
TL	X_{TL}= 65 Ω ($\bar{Z}_{\text{TL}} = jX_{\text{TL}}$)
TF2	A_n=30 MVA, $V_{n,1}$=220 kV, $V_{n,2}$=20 kV, $V_{sc,\%}$=10%
L	P_L=24 MW, Q_L=15 MVar, V_L=19 kV

Q1/ Compute the voltage magnitudes on the primary sides of the two transformers and at the internal bus of the synchronous machine.

[A1]

3.2 Parallel Transformers

Consider two transformers (TF1, TF2) connected in parallel as shown in Fig. 5, whose primary sides are connected to a medium-voltage distribution grid (G). The electrical parameters of the transformers are listed in Tab. 2.

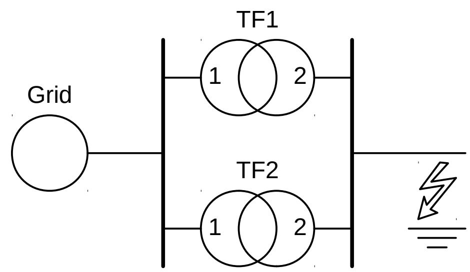


Figure 5: Short-circuit analysis of two parallel transformers.

Table 2: Electrical parameters of the two-transformer setup.

Component	Parameters
G	S_{sc}=348 MVA, V_n=20 kV ($\bar{Z}_\text{G} = jX_\text{G}$)
TF1	A_n=1000 kVA, $V_{n,1}$=20 kV, $V_{n,2}$=0.4 kV,
	$V_{sc,\%}$=5%, $\cos\phi_{sc}$=0.22
TF2	A_n=400 kVA, $V_{n,1}$=20 kV, $V_{n,2}$=0.4 kV,
	$V_{sc,\%}$=5%, $\cos\phi_{sc}$=0.22

Q2/ Compute the magnitude of the total short-circuit current on the secondary side of the transformers.

[A2]